Inverse cascade and intermittency of passive scalar in one-dimensional smooth flow

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Random advection of a Lagrangian tracer scalar field $\theta(t,x)$ by a one-dimensional, spatially smooth and short-correlated in time velocity field is considered. Scalar fluctuations are maintained by a source concentrated at the integral scale L. The statistical properties of both scalar differences and the dissipation field are analytically determined, exploiting the dynamical formulation of the model. The Gaussian statistics known to be present at small scales for incompressible velocity fields emerges here at large scales ($x \gg L$). These scales are shown to be excited by an inverse cascade of θ^2 and the probability distribution function (PDF) of the corresponding scalar differences to approach the Gaussian form, as larger and larger scales are considered. Small-scale ($x \ll L$) statistics is shown to be strongly non-Gaussian. A collapse of scaling exponents for scalar structure functions takes place: Moments of order $p \geq 1$ all scale linearly, independently of the order p. Smooth scaling x^p is found for $-1<p<1$. Tails of the scalar difference PDF are exponential, while at the center a cusped shape tends to develop when smaller and smaller ratios x/L are considered. The same tendency is present for the scalar gradient PDF with respect to the inverse of the Péclet number (the pumping-to-diffusion scale ratio). The tails of the latter PDF are, however, much more extended, decaying as a stretched exponential of exponent 2/3, smaller than unity. This slower decay is physically associated with the strong fluctuations of the dynamical dissipative scale. [S1063-651X(97)08311-6]

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I. INTRODUCTION

Small-scale statistics of a passive scalar advected by a large-scale incompressible velocity field is an old problem in turbulence theory [1,2]. Scalar fluctuations are maintained by a large-scale forcing, with typical scale L. According to the classical picture [3] of direct cascade of the scalar, the injected scalar is mainly transferred downscale to the convective interval and then to the dissipative region. For smooth velocities, the statistics of this scalar transfer can be analyzed systematically and has been characterized in much detail in [2,4,5]. The core of the one-point scalar probability distribution function (PDF) is Gaussian with variance $O(\ln \mathrm{Pe})$, where the Péclet number Pe is very large [5]. Far tails of the PDF decay exponentially [4,5]. The physical key ingredient at the basis of these results is that material lines are stretched, i.e., the maximum Lyapunov exponent $\overline{\lambda}$ for particle separation is positive. Typical trajectories will therefore be exponentially stretched and dynamically contracted trajectories are so rare that they can only affect the extreme tails of the statistics. On the other hand, much interest has been directed recently at the Kraichnan model [6] for its intermittent scaling behavior [7–9]. The picture emerging there is that dynamically contracted Lagrangian trajectories play a crucial role for structure functions scaling exponents ζ_p and thus for intermittency. The constant asymptotic behavior of ζ_p for large orders p found in [10] comes, for example, from the most contracting possible trajectories and the value ζ_∞ from nontrivial fluctuations of the degree of freedom constrained by incompressibility to still be dynamically stretched.

Previous remarks have led us to investigate scalar transport in a smooth compressible flow. The motivation is that compressibility might slow down Lagrangian separations and thus lead to nontrivial scaling and intermittency properties. These can then be analyzed systematically, using techniques specific for smooth velocities. Positive Lyapunov exponents are indeed characteristic of an isotropic, solenoidal flow [11,12], but this property might be lost when compressible flows are considered. It is, for example, known that for a compressible flow a substantial slowdown of long-time transport can take place (see [13]). Since trapping effects are amplified when the dimensionality of space is low, we have focused our attention on the one-dimensional case. More specifically, we have considered the smooth limit of the one-dimensional δ-correlated-in-time model recently introduced in [14]. In the absence of pumping and dissipation, any function of the tracer $\theta(t,x)$ (say, temperature) is advected along Lagrangian trajectories and globally conserved on average (provided the velocity is temporarily fast or spatially smooth). Switching on the energy (θ^2) supply at the integral scale, one expects that a steady (or quasisteady, as discussed in Sec. II) distribution of the scalar is attained. The main question raised here is how trapping effects due to compressibility affect the redistribution of energy among the scales and the intermittency properties of the scalar field at the stationary state. To answer this question, we exploit the dynamical formulation of the model to calculate the statistical properties of θ both upscale and downscale, i.e., at scales smaller and larger than the integral scale L. Since the scalar is a tracer in the velocity field and the velocity is smooth, the problem reduces to studying Lagrangian separations statistics.

The major physical difference appearing with respect to the incompressible case is that the maximum Lyapunov exponent for Lagrangian separations is negative. This means that, along typical trajectories, distances are mostly contracted and the stretching process is strongly intermittent in time. As in [15], the second and higher powers of the distance $R(t)$ between Lagrangian trajectories grow exponentially, while its low-order positive moments decay exponentially. This is the dynamical origin of the major results found in this paper: inverse cascade and Gaussian statistics at large scales and extreme intermittency at small scales.

Scalar correlations are indeed essentially governed by the statistics of the time spent by particles at distances smaller than L. Consider then two particles initially separated by a distance $x \gg L$. Despite the fact that $x \gg L$ initially, the distance $R(t)$ will typically reduce to $O(L)$ for large times $\sim \ln(x/L)/|\lambda|$. The consequence is that even scales much larger than L are strongly excited and this is the dynamical hint of the inverse energy cascade. Moreover, for moments of order $n \ll \ln(x/L)$, relative fluctuations around the previous typical time are small and this leads to the Gaussian statistics of scalar difference PDF. On the contrary, the statistics at small scales $x \ll L$ is associated with the stretching process. The time to reach the integral scale L strongly fluctuates and this is reflected in the intermittency of both scalar differences and gradients.

Note that the inverse cascade of the scalar taking place here differs in one important respect from other known examples of inverse cascades (say, an energy cascade in two-dimensional (2D) Navier-Stokes turbulence [16] and the number of wave cascades in wave turbulence [17]): No flux of another integral of motion (such as enstrophy or wave energy) is present. The origin of the inverse cascade found here is purely dynamical and associated with trapping effects. An interesting consequence of the inverse cascade is that the equation for the velocity difference PDF, derived here exactly from the dynamics, coincides with the one without a dissipative anomaly (an operator product expansion, which may result in the anomaly, was proposed in [18] in the context of the Burgers turbulence; see also [19] for possible extensions to the passive scalar turbulence). The absence of an anomaly is indeed a consequence of the inverse cascade, preventing the rare trajectories emerging from the dissipative range from affecting the convective range behavior.

Strong downscale intermittency emerges all over the quantities calculated in Secs. IV–VI. Moments of scalar differences of order $n \geq 1$ all scale linearly with x, independently of the order. This collapse of exponents carries over to the dissipation field ϵ, which has all its integer moments scaling with the same power of the Péclet number. Smooth scaling is observed for low-order moments of both scalar differences and dissipation. Very large fluctuations of these two quantities behave, however, quite differently. The scalar difference PDF has indeed a Lorentzian shape for values smaller than unity and exponential tails. The tails of the dissipation field PDF are, on the contrary, stretched exponentials with exponent 1/3 (and not 1/2). The additional probability for these strong events comes from fluctuations of the dynamical dissipative scale. A comparison of the nth moment of the dissipation field with the $2n$th moment of the scalar differences establishes indeed the effective viscous scale. This appears to be a strongly fluctuating quantity, growing factorially with n. (An essential enhancement of the dissipative scale was observed also in [20], where the tail of the scalar PDF was studied for the incompressible case by an instanton technique.) This factorial dependence is the cause of the 1/3 stretched exponential.

Even though the explicit calculations are quite lengthy, the underlying technical ideas, which make the analytical calculations doable, are simple to explain. Since the scalar is passive, its nth-order correlation function can be presented in terms of a matrix element of an auxiliary quantum mechanics. There are then two important steps in the evaluation of such matrix elements (and thus of the scalar PDFs). First, smoothness of the velocity field (stretchings and compressions are uniform on all the particles) allows us to reduce (Sec. III and Appendix A) the multiparticle problem to a one-parameter problem. The fact that the Lyapunov exponent is negative clearly emerges in this procedure. This leads, after the very direct calculations of Secs. IV and V, to a compact expression for the PDF of the scalar differences in the convective interval. Here the temporal dynamics of the fluctuating degree of freedom is local, while the locality is lost in the dissipative range. The second step then comes into play. To describe the convection-diffusion interplay in Sec. VI and Appendix B, we use a scale separation procedure. The temporarily nonlocal (diffusive) and local (convective) dynamics of the fluctuating degree of freedom are well separated by time t_0, $1 \ll t_0 \ll \ln[\mathrm{Pe}]$, if the Péclet number Pe is large. The independence of the resulting average (say, the gradient's PDF) over both local and nonlocal domains on t_0 and the smallness of the neglected terms with respect to the inverse Péclet number justifies the scale separation procedure.

The plan of the paper is as follows. In Sec. II the one-dimensional passive scalar model is briefly recalled, its relevant time and length scales are discussed, and the inverse cascade issue is explained from consideration of a scalar pair-correlation function. The dynamical formulation associated with the passive scalar equation is the subject of Sec. III and Appendix A. The latter is based on the Martin-Siggia-Rose formalism, while the former is in terms of the particle formalism. A key point arising in both procedures is that compressibility couples the dynamics with a global mode. This mode must then be taken into account in order to get the dynamical formulations. Multipoint correlation functions of scalar gradients are discussed in Sec. IV. The scalar difference PDF is described in Sec. V, where behaviors upscale and downscale with respect to L are considered in Secs. V A and V B, respectively. To describe the advection-diffusion interplay, we develop a scale separation procedure in Sec. VI, which is devoted to the PDF of dissipation. We use the scale separation procedure also in Appendix B to study the question of how the steady regime for the pair-correlation function of the gradients (discussed in Sec. II) forms. Section VII is reserved for conclusions and a discussion of questions that may be of a general relevance for other problems in turbulence and physics of disordered systems.

II. MODEL. THE PAIR-CORRELATION FUNCTION AND INVERSE CASCADE

Our aim here will be, first, to briefly recall the equations of the one-dimensional model introduced in [4], and then, solving the equation for scalar pair correlation function, show the effects of compressibility on the redistribution of energy among the scales and how the inverse cascade of the scalar works.

The advection-diffusion equation governing the evolution of the passive scalar $\theta(t,x)$ is

$$\partial_t\theta + u\partial_x\theta = \kappa\partial_x^2\theta + f. \tag{1}$$

[Note that there are two types of passive fields for compressible flow. Lagrangian tracers, like entropy or temperature (provided pressure is slowly varying in space and time), are conserved along Lagrangian trajectories in the absence of diffusion and pumping. The local maxima of the field do not grow in the absence of pumping. Concentration fields, e.g., of a pollutant, are, on the contrary, only globally conserved and their maxima can be amplified. The equations for the two types of fields differ by the position of the space derivative in the velocity term. In our one-dimensional case, the two possibilities correspond to the θ and ω fields, respectively.] The velocity field u and the force f are both assumed to be Gaussian and δ-correlated in time. The force has the correlation function

$$\langle f(t,x)f(t',x')\rangle = \delta(t-t')\chi\left(\frac{x-x'}{L}\right) \tag{2}$$

spatially concentrated at the integral scale L. The velocity has zero average and the correlation function

$$\langle u(t,x)u(t',x')\rangle = [V_0 - S(x-x')]\delta(t-t') \quad \text{with} \quad S(x)=|x|^{2-\gamma}. \tag{3}$$

The specific smooth case considered here corresponds to $\gamma=0$. The scaling behavior of the structure function S persists up to the infrared cutoff L_u, the largest scale in our problem. The scale-independent part of the velocity correlation function V_0 is estimated by the infrared cutoff of the velocity field squared $\sim L_u^2$. For scales larger than L_u, velocity correlations decay to zero, i.e., the structure function saturates to the constant value V_0. The equation of motion for the gradient field $\omega=\partial_x\theta$ immediately follows from Eq. (1):

$$\partial_t\omega + \partial_x(u\omega) = \kappa\partial_x^2\omega + \partial_x f. \tag{4}$$

The δ correlation in time of both the velocity and the forcing allows us to derive closed equations of motion for equal-time correlation functions [2]. It is easy, e.g., using Gaussian integration by parts (see [21]), to derive the equation for the second-order correlations $\Omega(t,x)=\langle\omega(t,x)\omega(t,0)\rangle$ and $F(t,x)=\langle\theta(t,x)\theta(t,0)\rangle$,

$$\partial_t\Omega = \partial_x^2\{[S(x)+2\kappa]\Omega\} - \partial_x^2\chi, \tag{5}$$

$$\partial_t F = [S(x)+2\kappa]\partial_x^2 F + \chi. \tag{6}$$

It is convenient to consider first Eq. (5) and then recover the correlations of the scalar by integration. From the very definition of the correlation function, it follows that the solution of Eq. (5) is even in x. Looking for a stationary solution, there is another boundary condition needed to fix the integration constants. This comes from the dynamics. Let us indeed consider the situation when the system starts from rest ($\theta\equiv 0$). At any subsequent time, the solution satisfies $\int\Omega\, dx=0$, where the integral is taken over all the space. Note that such a condition is consistent and actually dictated by the dynamics: The integral of the correlation function of the gradients is conserved in the evolution on account of the double space derivative on the right-hand side of Eq. (5). It is then easy to find the stationary solution of Eq. (5),

$$\Omega(x) = \frac{\chi(x/L)}{2\kappa+S(x)} - \frac{C}{2\kappa+S(x)} \quad \text{where} \quad C = \frac{\int \chi/(2\kappa+S)}{\int 1/(2\kappa+S)} \tag{7}$$

[see Appendix B for a dynamical derivation of Eq. (7)]. Expression (7) is very illuminating for several reasons. Let us first consider the case where no infrared cutoff L_u is present. The integral $\int 1/(2\kappa+S)$ is then convergent and, for small molecular diffusivity, varies as $1/\sqrt{\kappa}$. For the sake of concreteness, let us specifically consider here a forcing that is regular at the origin. The dominant terms for distances much smaller than the integral scale L are then

$$\Omega(x) = \frac{|\chi''(0)|}{2(2\kappa+S)L^2}[aL\sqrt{\kappa} - x^2], \tag{8}$$

where a is constant $O(1)$, dependent on the detailed form of the pumping. The first and the second term dominate, respectively, at scales smaller and larger than $L/\sqrt{\text{Pe}}$, where the Péclet number $\text{Pe}\equiv L/\sqrt{2\kappa}$ is supposed to be very large. Note that this scale is still much larger than the dissipative scale L/Pe. The most interesting aspect of Eq. (8) is that the dissipation $\kappa\Omega(0)$ vanishes as 1/Pe, i.e., there is no direct cascade. The energy is actually transferred upscale by an inverse cascade, as also emerges from the behavior of scalar correlations. [It is important to mention that, in the absence of pumping and dissipation, the average over the velocity of any function of the scalar $f(\theta)$ is conserved, although the average over all the space of $f(\theta)$ itself is not conserved in the particular realizations.] Let us indeed insert into Eq. (6) the correlation $\langle\theta(t,x_1)\theta(t,x_2)\rangle = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\Omega(y_1-y_2)$ with the ω-correlation function having expression (7). It is easily checked that the energy satisfies $\partial_t\langle\theta^2\rangle = C$, i.e., it grows linearly with time. Note that the growing mode is constant in space and thus disappears when differences or gradients are considered. The effect of the advective term $u\partial_x\theta$ is therefore to transfer energy upward in scale. Since dissipation is quadratic in wave number, the energy on large scales cannot practically be dissipated and it is accumulated. This is at the origin of the linear growth in time of $\langle\theta^2\rangle$. $C\neq 0$ corresponds to the inverse cascade, which therefore holds generically for any scaling exponent $\gamma<1$ in Eq. (3) (if the source function χ is not exceptional).

Let us now introduce an infrared cutoff L_u in the velocity field. Since S saturates to a constant, it is clear now that the integral $\int 1/(2\kappa+S)$ diverges. The constant C in Eq. (7) must then vanish. In the presence of a finite cutoff L_u, there will then be a very long, intermediate-time asymptotic where for scales $\ll L_u$ the behavior without cutoff is observed.

However, after a very long time $\mathcal{T}_{L_u}\sim(L_u/\kappa)^2$ the dynamics changes: The inverse cascade stops, $\langle\theta^2\rangle$ saturates to a finite value, the system starts dissipating a finite amount of energy in the limit of large Pe, and the correlation function $\Omega(x)$ tends to the solution (7) with $C=0$. The finite contribution $-\int\chi/(2\kappa+S)$ needed to ensure the zero integral condition comes from a strip of negative values that becomes more and more extended with time and whose amplitude tends to vanish. More details on the dynamics of the pair-correlation function of the scalar gradients at infinite times in the presence of an infrared cutoff L_u may be found in Appendix B.

III. DYNAMICS IN PARTICLE (LAGRANGIAN) FORMALISM

In this section we shall discuss the dynamical formulation of the equations of motion. The goal is the same as in [5,22]: to reduce the calculation of simultaneous scalar statistics to averaging of functionals of the random-in-time strain-vorticity matrix. This reduction is crucially based on the fact that the velocity field is smooth ($\gamma=0$) and can be performed for any smooth flow, independently of its compressibility and the dimension of space. In the specific 1D case, no matrices are obviously involved and one is left with averaging of a single scalar field. Compressibility, however, makes the derivation slightly more involved and some care must be taken in the ordering of the advective term. This emerges, in particular in the nonvanishing average of σ in the weight (21) for the Lagrangian trajectories (22) and nontrivial Jacobian of the transformation from the Eulerian frame to the one comoving with a fluid particle. The dynamical formulation is derived here using particle formalism. The equivalent derivation using Martin-Siggia-Rose field formalism is reserved for the Appendix A.

Equation (1) for the passive scalar can be presented in the form

$$\theta(T;x)=\int_0^T T\exp\left[\int_t^T\hat{\mathcal{P}}(t';x)dt'\right]\phi(t;x)dt$$
$$=\int_0^T dt\int dy\;\Psi(t,T;x,y)\phi(t;y), \tag{9}$$

where it was supposed that no pumping was supplied at negative times. In Eq. (9) the operator $\mathcal{P}(t;x)\equiv-u(t;x)\partial_x+\kappa\partial_x^2$, the time-ordered exponential is denoted by $T\exp$, and the function Ψ can be expressed by using Lagrangian trajectories as

$$\Psi(t,T;y,x)=\int_{\rho(t)=y}^{\rho(T)=x}\mathcal{D}\rho\;\exp\left[-\int_t^T(\dot{\rho}-u)^2/4\kappa\right]. \tag{10}$$

This formula expresses the simple fact that Lagrangian trajectories are fixed by the velocity u and smoothed by the molecular diffusivity κ. One can express Eq. (10) in the more convenient Hamiltonian form

$$\Psi(t,T;x,y)\equiv\int_{\rho_N=x}^{\rho_0=y}\prod_{n=0}^{N-1}dp_n\prod_{n=1}^{N-1}d\rho_n\;\exp\left[\Delta\sum_{n=1}^{N}\left(\frac{1}{\Delta}p_{n-1}(\rho_n-\rho_{n-1})-p_{n-1}u(t_n;\rho_n)+\kappa p_{n-1}^2\right)\right], \tag{11}$$

$$\rho_n=\rho(t_n=t+n\Delta),\quad p_n=p(t+n\Delta),\quad \Delta\equiv\frac{T-t}{N-1}\to0, \tag{12}$$

where the momenta integrations (dp_k) run along the imaginary axis and regularizations have been specified. Using the property that both velocity and pumping are Gaussian and have correlation functions (3) and (2), we can easily perform the averages in the $2n$th simultaneous product of the scalar field $\theta(T;x)$. We thus obtain

$$F(T;x_1,\ldots,x_{2n})\equiv\langle\theta(T,x_1)\cdots\theta(T,x_{2n})\rangle$$
$$=\int_0^T dt\int\prod_{i=1}^{2n}dy_i\mathcal{R}(T-t;x_i,y_i)$$
$$\times[F(t;y_1,\ldots,y_{2n-2})\chi(y_{2n}-y_{2n-1})$$
$$+(\text{permutations}), \tag{13}$$

where the eddy-diffusivity resolvent is defined as

$$\mathcal{R}(T;x_i,y_i)\equiv\prod_{i=1}^{2n}\langle\Psi(T,0;x_i,y_i)\rangle$$
$$=\int_{\rho_i(0)=x_i}^{\rho_i(T)=y_i}\mathcal{D}\rho_i(t)\mathcal{D}p_i(t)\exp\left(\int_0^T dt[p_i\dot{\rho}_i\right.$$
$$\left.-\tfrac{1}{2}p_i(\rho_i-\rho_j)^2p_j+\kappa p_i^2]\right). \tag{14}$$

Here one inverses the direction of time in comparison to Eq. (11) ($t\to T-t$) and thus the convective term ($p^2\rho^2$) is regularized in a way such that its ρ-dependent part is retarded with respect to the p-dependent part. The Hubbard-Stratonovich transformation of the diffusive term gives

$$\mathcal{R}(T;x_i,y_i)=\int_{\rho_i(0)=x_i}^{\rho_i(T)=y_i}\mathcal{D}\rho_i(t)\mathcal{D}p_i(t)\mathcal{D}\xi_i(t)\exp[-\mathcal{S}-\mathcal{S}_\xi], \tag{15}$$

$$\mathcal{S}=\int_0^T dt[-p_i\dot{\rho}_i+\tfrac{1}{2}p_i(\rho_i-\rho_j)^2p_j-p_i\xi_i],$$

$$\mathcal{S}_\xi=\frac{1}{4\kappa}\int_0^T \xi_i^2 dt. \tag{16}$$

In the integration in Eq. (13) of the resolvent, the correlation function F and the pumping correlation χ appear. Both depend on the difference of the coordinates only. This means that one can simply integrate with respect to collective variables, say, $\rho=\Sigma\rho_i$ and the collective momentum $P\equiv\Sigma_i^{2n}p_i/2n$. To get rid of the superfluous degrees of freedom, let us move from the old set of variables $\{\rho_1,\ldots,\rho_{2n};p_1,\ldots,p_{2n}\}$ to the new one $\{\rho,\tilde{\rho}_2,\ldots,\tilde{\rho}_{2n};P,\tilde{p}_2,\ldots,\tilde{p}_{2n}\}$. The new momenta $\tilde{p}_i=p_i-P$ are considered in the system comoving with the ''center of mass'' and the positions $\tilde{\rho}_i=\rho_i-\rho_1$ are with respect to one, e.g., ρ_1, taken as reference. The action S can then be decomposed as $\mathcal{S}=\mathcal{S}_{col}+\tilde{S}$, where

$$\begin{aligned}\mathcal{S}_{col}\equiv\int_0^T dt\Bigg[&-P^2\left(\sum_{i>1}\tilde{\rho}_i\right)^2+2nP^2\left(\sum_{i>1}\tilde{\rho}_i^{\,2}\right)\\&+2nP\left(\sum_{i>1}\tilde{p}_i\tilde{\rho}_i^{\,2}\right)-2P\left(\sum_{i>1}\tilde{p}_i\tilde{\rho}_i\right)\left(\sum_{j>1}\tilde{\rho}_j\right)-P\dot{\rho}\\&-P\sum_{i=1}\xi_i\Bigg],\end{aligned} \tag{17}$$

$$\tilde{\mathcal{S}}\equiv\int_0^T dt\left[-\left(\sum_{i>1}\tilde{p}_i\tilde{\rho}_i\right)^2-\sum_{i>1}\tilde{p}_i\dot{\tilde{\rho}}_i-\sum_{i>1}\tilde{p}_i(\xi_i-\xi_1)\right]. \tag{18}$$

Let us now consider the integral $\int\Pi_{i=1}^{2n}dy_i\mathcal{R}(T;x_i,y_i)f(y_i)$, where, as in Eq. (13), f is a function of differences only $f(y_i)=f(y_i-y)$. Collective degrees of freedom ρ and P are easily integrated. The principal point for this integration is the absence of a ''potential'' ρ dependence on the action $\mathcal{S}$. The integral is then reduced to $\int\Pi_{i>1}^{2n}d\tilde{y}_i\mathcal{R}(T;\tilde{x}_i,\tilde{y}_i)f(\tilde{y}_i+y_1)$, where the effective resolvent

$$\begin{aligned}\tilde{\mathcal{R}}(T;\tilde{x}_i,\tilde{y}_i)=&\int_{\tilde{\rho}_i(0)=\tilde{x}_i}^{\tilde{\rho}_i(T)=\tilde{y}_i}\mathcal{D}\tilde{\rho}(t)\mathcal{D}\xi_i(t)\mathcal{D}\tilde{\sigma}(t)\exp[-\mathcal{S}_\sigma-\mathcal{S}_\xi]\\&\times\prod_{m,i}\delta\left(\frac{\rho_i^{(m)}-\rho_i^{(m-1)}}{\Delta}-\tilde{\sigma}^{(m)}\rho_i^{(m-1)}\right.\\&\left.-\xi_i^{(m)}+\xi_1^{(m)}\right)\end{aligned} \tag{19}$$

and the measure of averaging $\mathcal{S}_\sigma=\Sigma_m[\tilde{\sigma}^{(m)}]^2/4$. In order to derive Eq. (19), we have decomposed the quadratic over $\tilde{p}\tilde{\rho}$ term by means of Hubbard-Stratonovich trick, introducing an additional collective integration over $\tilde{\sigma}$. The integration with respect to momentum $\tilde{p}_i$ is already performed in the last line of Eq. (19) (the effective action appears to be linear in $\tilde{p}$ in the result of the Hubbard-Stratonovich transformation). The continuous versions of the equation under the δ-function sign in Eq. (19) and the measure of averaging over $\tilde{\sigma}$ are

$$\dot{\tilde{\rho}}_i=\sigma\tilde{\rho}_i+\xi_i-\xi_1,\quad \sigma\equiv\tilde{\sigma}-1, \tag{20}$$

$$\mathcal{S}_\sigma=\frac{1}{4}\int_0^T dt[\sigma+1]^2. \tag{21}$$

[By continuous version we mean, in particular, a symmetrical smearing of the temporal δ function in the σ-field correlation for a small but finite (which is still much larger than the temporal slice Δ) width.] To see the relation between Eq. (19) and Eqs. (20) and (21) one can check in particular that both discrete and continuous versions give $\langle\rho_i(t)\rangle=\rho_i(0)$. The negativity of the Lyapunov exponent $\lambda\equiv\lim_{t\to\infty}\{\ln[W(t)/t]\}$ follows from Eq. (21). The formal solution of Eq. (20) is

$$\tilde{\rho}_i(t)=W(t)(x_i-x_1)+W(t)\int_0^t dt'\,W^{-1}(t')(\xi_i-\xi_1),$$

$$W(t)\equiv\exp\left[\int_0^t dt'\,\sigma(t')\right]. \tag{22}$$

It is finally easy to recalculate the $2n$-particle correlation function of the scalar from the $2n$-particle resolvent

$$\begin{aligned}F(T;x_1,\ldots,x_{2n})&\\=\langle&\Xi[T;\{\sigma(t)\};x_1-x_2]\cdots\Xi[T;\{\sigma(t)\};x_{2n-1}-x_{2n}]\\&+(\text{permutations})\rangle_\sigma,\end{aligned} \tag{23}$$

$$\begin{aligned}\Xi[T;\{\sigma(t)\};x_i-x_j]\equiv\int_0^T dt\Bigg\langle\chi\Bigg[&W(t)\frac{x_i-x_j}{L}\\&+W(t)\int_0^t dt'\,W^{-1}(t')\,\frac{\xi_i-\xi_j}{L}\Bigg]\Bigg\rangle_{\xi_{i,j}},\end{aligned} \tag{24}$$

where averages over $\sigma(t)$ and $\xi_i(t)$ are fixed by the measures $\exp[-\mathcal{S}_\sigma]$ and $\exp[-\mathcal{S}_\xi]$ defined in Eqs. (21) and (16), respectively.

IV. CORRELATION FUNCTIONS OF SCALAR GRADIENTS IN THE CONVECTIVE INTERVAL

Using the results of the preceding section, the dynamical expression of correlation functions of the scalar gradient $\omega(t,x)=\partial_x\theta(t,x)$ can be simply found differentiating Eq. (23) with respect to all spatial arguments. Here we shall be interested in the behavior of these simultaneous correlation functions for distances such that molecular diffusivity can be neglected. We first derive a general formula valid for an arbitrary form of forcing correlation and then treat more specifically the case of an exponential pumping. The resulting expression shows that the ratio between the irreducible and the reducible contributions to the $2n$th correlation function grows as $(L/x)^{n-1}$. This evidences both the non-Gaussian nature of the field and the fact that it increases going downscale in the convective interval. The same phenomenon takes place for generic forms of the pumping. The discussion of the range of scales where these considerations hold is postponed to the end of the section.

It follows from Eq. (23) that scalar gradients correlation functions are given by

$$\langle \omega(T,x_1)\cdots\omega(T,x_{2n})\rangle = \frac{(-1)^n}{L^{2n}}\left\langle \left[\int_0^T dt\ W^2(t)\chi''\left(W(t)\frac{x_1-x_2}{L}\right)\right]\cdots\left[\int_0^T dt\ W^2(t)\chi''\left(W(t)\frac{x_{2n}-x_{2n-1}}{L}\right)\right]\right\rangle + (\text{permutations}), \tag{25}$$

where molecular diffusion effects have been neglected. The only averaging left in Eq. (25) is then with respect to the σ statistics defined by Eq. (21). A possible way of performing this average is to introduce the auxiliary object

$$A(s_{1,2},\ldots,s_{2n,2n-1}) = \left\langle \exp\left\{\int_0^T W^2(t)\left(s_{1,2}\chi''\left[W(t)\frac{x_1-x_2}{L}\right] + \cdots + s_{2n,2n-1}\chi''\left[W(t)\frac{x_{2n}-x_{2n-1}}{L}\right]\right)dt\right\}\right\rangle. \tag{26}$$

Differentiating Eq. (26) over s variables, Eq. (25) is clearly reproduced up to the sign and the L-dependent prefactor. Inserting the weight (21) into Eq. (26), one can easily recognize the path-integral structure associated with the time evolution of the quantum-mechanical Schrödinger equation, having Hamiltonian $\hat{H} = -\partial_\eta^2 - U(\exp\,\eta)\exp(2\eta)$. The potential U appearing in the Hamiltonian is

$$U(y) \equiv s_{1,2}\chi''\left[y\frac{x_1-x_2}{L}\right] + \cdots + s_{2n,2n-1}\chi''\left[y\frac{x_{2n}-x_{2n-1}}{L}\right] \tag{27}$$

and the space variable η is defined as $\eta = \int_0^t dt'\,\sigma(t')$. Using standard notation for quantum-mechanics matrix elements, it is easy to check that Eq. (26) can be presented as

$$A = \exp[-T/4]\langle \delta(\eta)|e^{-T\hat{H}}|e^{-\eta/2}\rangle = [e^{-T/4}\Pi(T;\eta)]_{\eta=0}, \tag{28}$$

where the wave function $\Pi(T;\eta)$ satisfies $(\partial_T - \hat{H})\Pi = 0$ and the initial condition is $\Pi(0;\eta) = e^{-\eta/2}$. We can now remark that the potential part of the Hamiltonian vanishes at $\eta \to -\infty$, while the initial condition does not. The resulting asymptotic behavior at large times T will then be

$$\Pi(T;\eta=\ln[y]) \xrightarrow[T\to\infty]{} e^{T/4}\frac{\Pi_0}{\sqrt{y}}, \quad [\partial_y^2 + U(y)]\Pi_0(y) = 0. \tag{29}$$

A new variable $y = \exp\,\eta$ has been introduced. The boundary conditions for the spatial part Π_0 are easily derived from those for Π: Π_0 should tend to unity for $y \to 0$ and the ratio $\Pi_0/\sqrt{y}$ should vanish for y tending to infinity. It follows from Eq. (26) that the auxiliary object A is simply the function Π_0 calculated at $y=1$.

Derivatives of A at $s=0$ are needed for the calculation of gradients correlations (25). It is then convenient to present the solution of Eq. (29) in the form of an expansion with respect to the potential U:

$$A(s_{1,2},\ldots,s_{2n,2n-1}) = \sum_{k=0}^{\infty} \int_0^1 dy_1 \int_{y_1}^{\infty} dy_2 U[y_2] \int_0^{y_2} dy_3 \int_{y_3}^{\infty} dy_4 U[y_4] \cdots \int_0^{y_{2k-2}} dy_{2k-1} \int_{y_{2k-1}}^{\infty} dy_{2k} U[y_{2k}]. \tag{30}$$

Only the nth-order term of the expansion in Eq. (30) actually contributes to the $2n$th-order scalar gradients correlation function. Its final expression reads

$$\langle \omega_1\cdots\omega_{2n}\rangle = \frac{(-1)^n}{L^{2n}} \sum \int_0^1 dy_1 \int_{y_1}^{\infty} dy_2 \chi''\left[y_2\frac{x_{k_1}-x_{k_2}}{L}\right] \int_0^{y_2} dy_3 \int_{y_3}^{\infty} dy_4 \chi''\left[y_4\frac{x_{k_3}-x_{k_4}}{L}\right] \cdots \int_0^{y_{2n-2}} dy_{2n-1} \times \int_{y_{2n-1}}^{\infty} dy_{2n}\chi''\left[y_{2n}\frac{x_{k_{2n}}-x_{k_{2n-1}}}{L}\right], \tag{31}$$

where summation is performed over all the splittings of the set $\{k_1,\ldots,k_{2n}\}$ into n ordered pairs.

Formula (31) holds for any pumping χ. Let us now specifically consider the case of the exponential pumping $\chi_e(z)=\exp[-|z|]$, where the integrals appearing in Eq. (31) can be easily performed. For the second-order correlation, we obtain that the dominant contribution for small distances x_{12}/L is $-1/x_{12}L$, in agreement with the solution (7). For the fourth-order correlation function, we find

$$\langle\omega(x_1,t)\omega(x_2,t)\omega(x_3,t)\omega(x_4,t)\rangle_e$$
$$=F_e(x_{12},x_{34})+F_e(x_{13},x_{24})+F_e(x_{14},x_{23}),$$
$$F_e(x,y)\simeq\frac{2}{xy(x+y)L}, \tag{32}$$

where $x_{ij}\equiv|x_i-x_j|$, the subscript e is intended to remind the reader that this specific expression holds for the exponential pumping, and only dominant terms in x_{ij}/L have been retained. Distinguishing between reducible (Gaussian) and irreducible contributions into Eq. (32), one observes that the irreducible part is $L/x\gg1$ times larger. More generally, if x denotes the typical distance between the various particles, i.e., $x_{ij}\sim x$, then $\langle\omega_1\cdots\omega_{2n}\rangle\sim1/(x^{2n-1}L)$. Both the correlation functions and the degree of non-Gaussian nature of the scalar gradient (ratio of the $2n$th moment to its reducible part) are then growing with x_{ij} going downscale.

Let us finally arrive at the range of validity of the previous convective arguments. The exponential form of the pumping is a special one since it is not regular at the origin. The first term of its expansion at small distances is linear and not quadratic. This affects the dependence on x of the second-order correlation. For a regular pumping, the dominant term would indeed be constant, as follows from Eq. (8). On the contrary, one can check using Eq. (31) that the dependence of correlations of order greater than or equal to 4 on x and L remains the same as for the exponential pumping. The regularity at the origin of the pumping also enters the range of scales where neglecting molecular diffusivity effects is allowed. Performing the small distance expansion, as in Eq. (8) for a regular pumping, a logarithmic correction proportional to ln Pe appears. The ultraviolet criterion of applicability of the previous convective considerations for the exponential pumping is then $x\gg\sqrt{\kappa}\,\ln[\mathrm{Pe}]$. For a pumping regular at the origin, the criterion is the same as in Eq. (8), i.e., $x\gg L/\sqrt{\mathrm{Pe}}$.

V. SCALING AND THE PDF OF SCALAR DIFFERENCES

Scalar structure functions $S_{2n}(x)=\langle[\theta(T,x)-\theta(T,0)]^{2n}\rangle$ can be easily expressed in terms of scalar correlation functions F by taking the appropriate combinations of them. A dynamical expression for S_{2n} can thus be derived directly from Eq. (24), obtained in Sec. III for the F's. This is not, however, a very practical procedure. Each of the contributions appearing in the sum giving the structure functions contains indeed the constant mode. As it was discussed in Sec. II and as it also appears from the dynamical expression (23), this mode grows linearly with the observation time T. It is just in the whole sum that these divergent contributions are canceled, thus leaving the time-independent final result for structure functions. It is then more convenient to restore structure functions directly from scalar gradients correlations as $S_{2n}(x)=\int_0^x dx_1\cdots\int_0^x dx_{2n}\langle\omega_1\cdots\omega_{2n}\rangle$. An important question is whether or not we can avoid taking dissipation explicitly into account in the calculation of S_{2n} in the convective interval. This means essentially asking whether it is enough to know just the convective expressions for gradients correlations or whether their whole behavior is needed. This point can be tested by simply taking the convective expressions for $\langle\omega_1\cdots\omega_{2n}\rangle$ found in Sec. IV and inserting them into the integral expression for S_{2n}. One can then check that all the integrals for any structure function are convergent on the ultraviolet and dominated by the infrared side of the convective range.

The expression for structure functions in the convective interval is then

$$S_{2n}(x)=(2n-1)!!n!2^n\int_0^{x/L}dy_1\int_{y_1}^{\infty}dy_2\frac{\chi[0]-\chi[y_2]}{y_2^2}\int_0^{y_2}dy_3\int_{y_3}^{\infty}dy_4\frac{\chi[0]-\chi[y_4]}{y_4^2}\cdots\int_0^{y_{2n-2}}dy_{2n-1}\times\int_{y_{2n-1}}^{\infty}dy_{2n}\frac{\chi[0]-\chi[y_{2n}]}{y_{2n}^2}, \tag{33}$$

where we have already performed the $2n$ integrals over the dx_i's. The whole set of equations (33) can be recast into the more compact equation

$$\left\{x^2\partial_x^2-\lambda^2\left[\chi(0)-\chi\left(\frac{x}{L}\right)\right]\right\}\mathcal{Z}\left(\frac{x}{L},\lambda\right)=0, \tag{34}$$

for the generating function $\mathcal{Z}(x,\lambda)=\langle\exp(-i\lambda\delta\theta_x)\rangle$ of scalar differences $\delta\theta_x$ taken at the scale x. From this very definition it follows that the function Z must tend to unity for vanishing x and, for the convergence of the integrals in Eq. (33), it should grow slower than linearly at infinity.

It is worth recalling that Eq. (34) was found as the result of an accurate dynamical procedure: We first calculated correlation functions of the scalar gradient for all points being separated, the resulting gradients correlation functions were then integrated to obtain structure functions, and finally the generating function for scalar differences was reconstructed from its moments. We thus avoided handling diffusion explicitly, paying for this the price of taking many particles into consideration. The closed differential equation (34) for the scalar difference generating function emerges as the result of this procedure. On the other hand, one could generally [also for the nonsmooth case, i.e., $\gamma \neq 0$ in Eq. (3)] derive the following unclosed Fokker-Planck equation for the generating function:

$$\{x^{2-\gamma}\partial_x^2 - \lambda^2[\chi(0)-\chi(x/L)]\}\mathcal{Z}(x/L,\lambda) = \kappa\langle[\partial_1^2\theta_1 - \partial_2^2\theta_2]\exp[\lambda(\theta_1-\theta_2)]\rangle. \tag{35}$$

This equation is simply obtained by averaging the equation of motion (1) for the scalar at two reference points. The smooth ($\gamma=0$) limit of Eq. (35) differs from Eq. (34) by the right-hand-side dissipative term. There is a general expectation that this term may remain finite even in the limit $\kappa \to 0$, thus providing a nonvanishing anomaly in the terminology of Polyakov [18]. Equation (34), derived microscopically without any conjecture, shows then the absence of an anomaly for the one-dimensional smooth flow. (We acknowledge Polyakov for directing our attention to this matter). Physically, the absence of an anomaly is associated with the vanishing direct flux of θ^2 in the limit of infinite Péclet numbers. This point, already appearing in Sec. II, will emerge even more clearly in the full analysis of the dissipation field in Sec. VI. In Secs. V A and V B solutions of Eq. (35) at $x \gg L$ and $x \ll L$ will be discussed.

A. Upscale interval

Let us first consider scales larger than the integral scale L. The asymptotic solution for the generating function, which can also be found from Eq. (35) by replacing $\chi(0)-\chi(x/L)$ by $\chi(0)$, is

$$\mathcal{Z} = \left(\frac{x}{L}\right)^{1/2-\sqrt{1+4\chi[0]\lambda^2}/2}. \tag{36}$$

The inverse Fourier transform of Eq. (36), calculated in the saddle-point manner (the large parameter is x/L or, equivalently, large values of $|\delta\theta_x|$), gives

$$\mathcal{P}^{\delta\theta_x}(y) = \frac{1}{2\sqrt{\pi\chi[0]\ln[x/L]}} \times \begin{cases} \exp\{-y^2/(4\chi[0]\ln[x/L])\}, & |y| \ll \ln[x/L] \\ \exp\{-|y|/(2\sqrt{\chi[0]})\}, & |y| \gg \ln[x/L]. \end{cases} \tag{37}$$

Thus the arguments for Gaussian statistics presented in [5] for the incompressible case indeed can be reversed and applied here for the upper interval. Structure functions of orders much less than $\ln[x/L]$ indeed scale logarithmically, the core of the scalar difference PDF is Gaussian, and the PDF's tail is exponential.

B. Downscale interval

We shall now obtain a general formula for the PDF at $y \ll 1$, no matter how y and x/L relate to each other, provided both of them are small. Indeed, replacing the χ-dependent potential by the first term of the expansion over x/L and solving the resulting differential equation with the same boundary conditions as before one gets the simple exponential form for the generating function $\mathcal{Z}(x/L,\lambda) = \exp(-\lambda\sqrt{\chi''[0]/2}x/L)$. The inverse Fourier transform produces the following Lorentzian expression for the PDF:

$$\mathcal{P}^{(\delta\theta_x)}(y) = \frac{1}{\pi}\frac{L}{x}\frac{1}{y^2+|\chi''[0]|x^2/(2L^2)}, \tag{38}$$

which is thus generally valid at $|y| \ll 1$. It results from Eq. (38) that the PDF is smooth in a small region around the origin $y=0$, where it can actually be expanded in y^2L^2/x^2. This region extends approximately up to x/L, where the second behavior in $1/y^2$ sets in. Note that the concavity of the PDF in this second region is upward and remain upward up to the small values x/L. It is just for very small values $|y| \ll x/L$ that the concavity is reversed downward. An experimental histogram of such a PDF would then look strongly cusped at the origin.

The tail of the PDF matching the Lorentzian (38) at $|y| \sim 1$ is exponential. To see this and generally to obtain an explicit analytic solution for the PDF in the whole domain of x and $\delta\theta_x$, let us consider the specific form of pumping

$$\chi_*\left(\frac{x}{L}\right) = \frac{1}{1+(x/L)^2}. \tag{39}$$

Making in Eq. (34) the change of variable $\cot[\varphi]=x/L$, the solution of the resulting equation can be expressed in terms of associated Legendre functions as

$$\mathcal{Z}_*(\cot[\varphi];\lambda) = \frac{2^{\nu-1/2}(\nu+1)\Gamma^2[(\nu+1)/2]P_{1/2}^{-\nu-1/2}(\cos[\varphi])}{\sqrt{\pi}\,\sin[\varphi]}, \tag{40}$$

where the upper index $\nu = -1/2+(1/2)\sqrt{1+4\lambda^2}$. The choice of the sign for the square root is such as to ensure that the generating function $Z(x,\lambda)$ grows at infinity slower than linearly. The notation $\mathcal{Z}_*$ in Eq. (40) is intended to stress that this explicit solution was obtained for the pumping (39). Note that Eq. (40) is particularly applicable for the upper interval discussed in Sec. V A. Considering Eq. (40) at $\varphi \ll 1$ and $\lambda \gg 1$, one indeed recovers Eq. (36). Using the integral representation (8.714) in [23] and the doubling formula for the Γ function, Eq. (40) can be presented in the integral form

$$\mathcal{Z}_*(\cot[\varphi];\lambda)=\frac{2\Gamma[(\nu+3)/2]}{\sqrt{\pi}\,\sin[\varphi]\Gamma[(\nu+2)/2]}\times\int_0^{\varphi}\cos[t]\left(\frac{\cos[t]-\cos[\varphi]}{\sin[\varphi]}\right)^{\nu}dt. \tag{41}$$

The integral representation (41) is useful since it clearly shows the analytic structure of $\mathcal{Z}_*$ with respect to λ. One observes in particular that $\mathcal{Z}_*$ is analytic in the whole upper semiplane, except on the line of imaginary λ from $i/2$ to $i\infty$. The path on the real axis in the inverse Fourier transform can be then deformed into the following one surrounding the cut: The left branch goes from $i\infty-0^+$ to $i/2-0^+$, bypasses $\lambda=i/2$ from below, and the right branch goes from $i/2+0^+$ to $i\infty+0^+$. The final result is

$$\mathcal{P}_*^{(\delta\theta_x)}(y)=\frac{1}{4\pi}\int_0^{\infty}\frac{q\,dq}{\sqrt{1+q^2}}\exp\left[-\frac{|y|}{2}\sqrt{(1+q^2)}\right]\times\mathcal{G}(\cot[\varphi];q), \tag{42}$$

where the function $-i\mathcal{G}$ is the difference between $\mathcal{Z}_*$ on the right and the left of the cut, i.e., $\nu\to-1/2\pm iq/2$, respectively. The general expression for $\mathcal{G}$ can be derived from the integral representation (41) as

$$\mathcal{G}(\cot[\varphi];q)=\frac{4}{\sqrt{\pi}\,\sin[\varphi]}\int_0^{\varphi}\frac{\cos[t]dt}{\sqrt{\cos[t]-\cos[\varphi]}}\,\mathrm{Re}\left[i\frac{\Gamma\left(\frac{5+iq}{4}\right)}{\Gamma\left(\frac{3+iq}{4}\right)}\left(\frac{\sin[\varphi]}{\cos[t]-\cos[\varphi]}\right)^{-iq/2}\right]. \tag{43}$$

The calculation of the PDF of scalar differences in the convective interval is now reduced to the evaluation of the asymptotic behavior of $\mathcal{G}$ in Eq. (43) and then to perform the integral (42). In the convective interval, $x/L=\cot[\varphi]\ll 1$, i.e., the angle φ is very close to $\pi/2$. The dominant expression of $\mathcal{G}$ in this region can be obtained by simply expanding directly in Eq. (43). Inserting this expansion into Eq. (42), we obtain

$$\mathcal{P}_*^{(\delta\theta_x)}(y)=\frac{\pi}{8}\frac{x}{L}\int_0^{\infty}\frac{q\sqrt{1+q^2}\,\sinh[\pi q/2]\exp\left[-\frac{|y|}{2}\sqrt{1+q^2}\right]dq}{|\Gamma[(3+iq)/4]|^4\cosh^2[\pi q/2]}\to\begin{cases}\dfrac{1}{\pi}\dfrac{x}{L}\dfrac{1}{y^2}, & 1\gg|y|\gg x/L\\ \sim\dfrac{x}{L}\exp[-|y|/2], & |y|\gg 1,\end{cases} \tag{44}$$

where, at $y\gg 1$, the prefactor algebraic in y has not been considered. Varying the pumping function $\chi(x/L)$, one can change the number behind $|y|$ in the exponential, but the exponential behavior itself will never change.

One may check that the PDF's asymptotic for the smallest values $|y|\ll x/L$ derived from Eqs. (40) and (41) is consistent with the general formula (38). Indeed, the respective large-q asymptote of $\mathcal{G}$ is

$$\mathcal{G}(\cot[\varphi];q)\sim 2\sqrt{\frac{q}{\pi\sin[\varphi]}}\int_0^{\varphi}\frac{\cos[t]}{\sqrt{\cos[t]-\cos[\varphi]}}\sin\left(\frac{q}{2}\ln\left[\frac{\sin[\varphi]}{\cos[t]-\cos[\varphi]}\right]-\frac{\pi}{4}\right). \tag{45}$$

Substituting Eq. (45) into Eq. (42) and performing first the integral in q and then the one in t, we obtain at $|y|\ll x/L\ll 1$

$$\mathcal{P}_*^{(\delta\theta_x)}(y)\sim\frac{1}{\sqrt{2}\pi}\int_0^{\varphi}\frac{\cos[t]dt}{\sqrt{\cos[t]-\cos[\varphi]}}\ln^{-3/2}\left[\frac{1}{\cos[t]-\cos[\varphi]}\right]\left(1-\frac{15}{8}\frac{y^2}{\ln^2\left[\frac{1}{\cos[t]-\cos[\varphi]}\right]}\right)\to\frac{1}{\pi}\frac{L}{x}\left(1-\frac{L^2y^2}{x^2}\right). \tag{46}$$

Expressions (44) and (46) are clearly in agreement with Eq. (38), valid for an arbitrary form of pumping.

Let us now derive the scaling behavior of scalar structure functions $\langle|\theta_1-\theta_2|^a\rangle$ at $x\ll L$. The scaling for orders $a>1$ is dominated by the behavior of the PDF at values of order unity. On the contrary, for $a<1$, the region in $1/y^2$ dominates. The resulting scaling behavior of structure functions is

$$\langle|\theta_1-\theta_2|^a\rangle\sim\begin{cases}\dfrac{x}{L} & \text{for } a\geq 1\\ \left(\dfrac{x}{L}\right)^a & \text{for } -1<a\leq 1.\end{cases} \tag{47}$$

For $a<-1$ the moments do not exist at all since the PDF is finite at the origin. We remark that Eq. (47) is exactly the same scaling as for velocity structure functions in the Burgers equation. In the present model the collapse of high-order exponents is associated with the uniformity in space of stretchings and compressions.

VI. PDF OF SCALAR DISSIPATION AND GRADIENTS

The nth-order moment of the dissipation field $\epsilon=\kappa(\partial_x\theta(t;x))^2$ can be obtained from the dynamical expression (24) as

$$\langle\epsilon^n\rangle=(2n-1)!!\,\kappa^n\langle(\partial_{x_i}\partial_{x_j}\Xi[T;\{\sigma(t)\};x_i-x_j])^n_{x_i=x_j}\rangle$$
$$\equiv(2n-1)!!\langle Q[T;\{\sigma(t)\}]^n\rangle_\sigma. \tag{48}$$

The average with respect to the noises ξ_j is easily performed in Fourier space, thus obtaining

$$Q\equiv\frac{1}{\mathrm{Pe}^2}\int_0^T dt\ W^2(t)G\left[\frac{W^2(t)}{\mathrm{Pe}^2}\int_0^t dt' W^{-2}(t')\right],$$

$$G(x^2)\equiv\frac{1}{4\pi}\int_{-\infty}^{\infty}dq\ q^2\chi_q\ \exp(-q^2x^2), \tag{49}$$

where $W(t)$ is defined in Eq. (24) and χ_q is the Fourier transform of the pumping correlation function $\chi(x)$. Let us recall that the Péclet number is defined as the pumping-to-diffusive scale ratio $\mathrm{Pe}\equiv L/\sqrt{2\kappa}$ and is supposed to be large. To obtain the stationary value $\langle\epsilon^n\rangle$, the limit $T\to\infty$ should be considered.

The average over σ with the Gaussian weight (21) has to be calculated in Eq. (48). This is very hard to do explicitly in the general case. We can, however, exploit the presence of the large parameter Pe to develop an asymptotic theory that captures the dominant terms in Eq. (48) with respect to Pe. The important point is that, when Pe is large, there are two very different time scales in the dynamics. For the Lagrangian trajectories relevant to Eq. (48), particles start very close to each other. The additive molecular noise term in the Langevin equation for Lagrangian trajectories is dominant at these distances and remains dominant as long as the particles do not separate by a distance comparable to the dissipative scale. This phase of the dynamics corresponds to the formation of the integral in the square brackets in Eq. (49) and takes place on times of order unity (not scaling with Pe). Once the particles have reached the dissipative scale and enter into the convective region, random multiplicative effects due to the velocity become dominant. Due to the multiplicative nature of the dynamics, the time to go from the dissipative scale to the integral scale varies as ln Pe. This phase is associated with the growth of the W^2 terms in Eq. (49). For large Péclet numbers, the two processes, formation of the integral in the square brackets and growth of W^2 terms in Eq. (49), are well separated in time. Let us then consider a time t_0 satisfying $1\ll t_0\ll\ln[\mathrm{Pe}]$. At the dominant order in the number Péclet, Q can be approximated as

$$Q[T;\{\sigma(t)\}]\approx\frac{\beta}{\mathrm{Pe}^2}\int_{t_0}^T dt\ \exp\left[2\int_{t_0}^t\sigma_>(t')dt'\right]$$
$$\times G\left[\alpha\frac{\exp\left[2\int_{t_0}^t\sigma_>(t')dt'\right]}{\mathrm{Pe}^2}\right], \tag{50}$$

where α and β are defined as

$$\alpha\equiv\exp\left[2\int_0^{t_0}\sigma(s)ds\right]\int_0^{t_0}dt'\ \exp\left[-2\int_0^{t'}\sigma(t'')dt''\right],$$

$$\beta\equiv\exp\left[2\int_0^{t_0}\sigma(t')dt'\right]. \tag{51}$$

In order to obtain Eq. (50) from the original expression (49), we have made the following two steps: $Q[t_0;\{\sigma(t)\}]$ has been neglected and the upper bound t in the integral over dt' for α has been replaced by t_0. Both steps are motivated by the time scales separation at large Péclet numbers. More precise conditions of validity of the approximation will be discussed later in the section. The moments $\langle Q^n\rangle$ can now be obtained by taking the nth power of Eq. (50) and averaging. The average over $\sigma(t)$ is decomposed in a small times part $\mathcal{D}\sigma_<\equiv\Pi_{t<t_0}d\sigma(t)$ and a large times part $\mathcal{D}\sigma_>\equiv\Pi_{t>t_0}d\sigma(t)$. The corresponding weights are simply obtained decomposing the integral over t in Eq. (21) as $\mathcal{S}_<=(1/4)\int_0^{t_0}[\sigma_<+1]^2$ and in $\mathcal{S}_>$ the integration runs from t_0 to T. The great advantage of Eq. (50) is that, in the long-time averaging, both α and β are just external parameters, depending neither on time t nor on $\sigma_>$. Once the average over $\sigma_>$ is performed, we are then left with a function of α and β. This is in turn averaged over $\sigma_<$, giving the final result $\langle Q^n\rangle$.

A compact way for averaging over long-time statistics is to introduce the Laplace transform of the PDF $\mathcal{P}_s^>\equiv\langle\exp[-sQ]\rangle_>$. It is indeed easy to recognize that its path integral coincides with a matrix element in quantum mechanics with Hamiltonian $\hat{H}=-\partial_\eta^2+s\beta\exp(2\eta)G[\alpha\exp(2\eta)/\mathrm{Pe}^2]/\mathrm{Pe}^2$. The space variable $\eta=\int_{t_0}^t\sigma(t')dt'$. Using standard quantum-mechanical notation, the expression for $\mathcal{P}_s^>$ can be presented as

$$\mathcal{P}_s^>\equiv\langle\exp[-sQ]\rangle_>$$
$$=\exp[-(T-t_0)/4]\langle\delta(\eta)|e^{-(T-t_0)\hat{H}_0}|e^{-\eta/2}\rangle$$
$$=[e^{-(T-t_0)/4}\Phi(T-t_0;\eta)]_{\eta=0}, \tag{52}$$

where $\Phi(t;\eta)$ satisfies $(\partial_t-\hat{H})\Phi=0$. The initial condition at $t=0$ for the wave function $\Phi(t;\eta)$ is $\exp(-\eta/2)$. Noting that the potential part of the Hamiltonian vanishes at $\eta\to-\infty$ and $\Phi(0;\eta)$ does not, we obtain the asymptotic behavior of Φ at long times

$$\Phi(t,\eta=\ln[y\mathrm{Pe}])\xrightarrow[t\to\infty]{}e^{t/4}\frac{\Phi_0(y)}{\sqrt{\mathrm{Pe}y}},$$

$$[\partial_y^2-s\beta G(\alpha y^2)]\Phi_0(y)=0. \tag{53}$$

The new variable $y=\exp(\eta)/\mathrm{Pe}$ has been introduced. The function $\Phi_0(y)$ should tend to unity for $y\to 0$ and $\Phi_0/\sqrt{y}$ should vanish for $y\to\infty$. It follows from Eq. (52) that $\mathcal{P}_s^>$ is simply the function Φ_0 calculated at $y=1/\mathrm{Pe}$.

The general way to attack Eq. (53) for an arbitrary form of pumping is to proceed as we have already done for Eq. (30) in Sec. IV. Starting with a constant unit solution, the term with s in Eq. (53) is treated perturbatively. A series in s is then obtained $\mathcal{P}_s^> = 1+\Sigma_{n=1}^{\infty} c_n(y) s^n$, with the c_n's having an expression similar to Eq. (31). The nth moment ϵ^n averaged over $\sigma_>$ coincides with $c_n(1/\mathrm{Pe})$ up to simple combinatorial factors. The final result is

$$\langle \epsilon^n\rangle_> = (2n-1)!!n!\left(\frac{\beta}{\alpha}\right)^{n-1/2}\frac{a_n\sqrt{\beta}}{\mathrm{Pe}}, \tag{54}$$

where the coefficients a_n are given by

$$a_n=\int_0^\infty dy_2 G[y_2^2]\int_0^{y_2} dy_3\int_{y_3}^\infty dy_4 G[y_4^2]\cdots\int_0^{y_{2n-2}} dy_{2n-1}\int_{y_{2n-1}}^\infty dy_{2n} G[y_{2n}^2]. \tag{55}$$

Equation (54), together with Eq. (61) allowing the calculation of small times averages, gives the expression of all integer moments of the dissipation field for a general form of pumping. It follows from Eq. (54) that all these moments scale with the same power of Pe. This shows that the strong intermittence evidenced in the analysis of scalar differences in the convective range comes down into the dissipative range. The analysis of the n dependence of the constants in Eq. (54) actually shows that the intermittence of the dissipation field is even stronger than for scalar differences. To this aim, it is convenient to restrict the analysis to a particular form of pumping, allowing us to proceed with explicit calculations. Specifically, let us consider $G^*[x^2]=\exp[-2x]$. The correlation function of the corresponding pumping has the Fourier transform $\chi_q^* = 4\sqrt{\pi}/q^4 \exp(-1/q^2)$. Equation (53) with G having the specific form G^* is solved in terms of the Bessel function I_0 as

$$\mathcal{P}_s^> = \frac{I_0(\sqrt{s\beta/\alpha}e^{-\sqrt{\alpha}/\mathrm{Pe}})}{I_0(\sqrt{s\beta/\alpha})} \xrightarrow[\mathrm{Pe}\gg 1]{} 1-\frac{\sqrt{s\beta}}{\mathrm{Pe}}\frac{I_1(\sqrt{s\beta/\alpha})}{I_0(\sqrt{s\beta/\alpha})}. \tag{56}$$

The advantage with respect to the general case with arbitrary form of pumping is clearly that $\mathcal{P}_s^>$ is now known explicitly and this will permit us to reconstruct the PDF of the dissipation field.

Having averaged over $\sigma_>$, we need now to take into account fluctuations at small times, i.e., average over $\sigma_<$. An important point is that, both in the general case (54) and in Eq. (56), we need to average quantities of the form $\sqrt{\beta}f(\alpha/\beta)$, with f arbitrary but having the property that it depends only on α/β. For our purposes, it is then appropriate to introduce the random variable $\mu=\alpha/\beta$ and consider its distribution function weighted with $\sqrt{\beta}$:

$$\mathcal{P}^<[\mu]\equiv\frac{\int \mathcal{D}\sigma_<\exp(-S_<)\,\delta\left(\mu-\frac{\alpha}{\beta}(\{\sigma_<\})\right)\sqrt{\beta}(\{\sigma_<\})}{\int \mathcal{D}\sigma_<\exp(-S_<)}. \tag{57}$$

It is again convenient for averaging to introduce the Laplace transform of the PDF $\mathcal{P}_s^< = \langle\exp(-s\mu)\rangle_<$. As in Eq. (52), its expression can be presented as the quantum-mechanical matrix element

$$\begin{aligned}\mathcal{P}_s^< &= \int_0^\infty e^{-s\mu}\mathcal{P}^<[\mu]d\mu = e^{-t_0/4}\langle\delta(\eta)|\exp[-t_0\hat{H}]|e^{\eta/2}\rangle\\ &=[e^{-t_0/4}\Psi(t_0;\eta)]_{\eta=0}.\end{aligned} \tag{58}$$

Here $\Psi(t;\eta)$ satisfies $(\partial_t-\hat{H})\Psi=0$, the Hamiltonian is $\hat{H}(\eta;s)=-\partial_\eta^2+s\exp(-2\eta)$, the space variable $\eta=\int_0^t\sigma$, and the initial condition for the wave function $\Psi(t;\eta)$ is $\exp(\eta/2)$. The asymptotic behavior at large times t can be obtained as in Eq. (53), noting that the potential part in $\hat{H}$ decreases at infinity and $\Psi(0;\eta)$ does not. It follows that

$$\Psi(t;\eta)\xrightarrow[t\to\infty]{} e^{t/4}\Psi_0(\eta)=e^{t/4}\exp[\eta/2-\sqrt{s}\exp(-\eta)], \tag{59}$$

where $\Psi_0(\eta)$ satisfies $(\hat{H}+1/4)\Psi_0=0$ and behaves as $\exp(\eta/2)$ for large η's. Requiring $\exp(t_0/4)\gg 1$, we can plug the asymptotic expression (59) into Eq. (58) and obtain $\mathcal{P}_s^<=\exp[-\sqrt{s}]$. The PDF of μ and the moments relevant for ϵ^n are easily restored as

$$\begin{aligned}\mathcal{P}^<(\mu)&=\frac{1}{2\pi i}\int_{0^+-i\infty}^{0^++i\infty} ds\mathcal{P}_s^< e^{s\mu}=\frac{1}{2\pi i}\int_{0^+-i\infty}^{0^++i\infty} ds e^{-\sqrt{s}}e^{s\mu}\\ &=\frac{1}{2\sqrt{\pi}\mu^{3/2}}\exp\left[-\frac{1}{4\mu}\right],\end{aligned} \tag{60}$$

$$\begin{aligned}\left\langle\left(\frac{\beta}{\alpha}\right)^{n-1/2}\sqrt{\beta}\right\rangle_< &= \int_0^\infty d\mu\ \mu^{-n+1/2}\mathcal{P}^<(\mu)\\ &=2^{2n-1}\frac{(n-1)!}{\sqrt{\pi}}.\end{aligned} \tag{61}$$

This expression can be used to calculate the moments appearing in Eq. (54). Note that expression (54), derived exploiting the time scale separation, coincides for the first moment $\langle\epsilon\rangle$ with the dominant order of the known solution derived in Sec. II:

$$\langle\epsilon\rangle=\frac{1}{\pi\mathrm{Pe}}\int_0^\infty\frac{[\chi(0)-\chi(x)]dx}{x^2+\mathrm{Pe}^{-2}}. \tag{62}$$

One step further can be made for the specific form of pumping $G^*[x^2]=\exp[-2x]$, allowing us to obtain the explicit solution (56). From Eq. (56) we obtain indeed the Laplace transform of the PDF for Q (averaged over both $\sigma_<$ and $\sigma_>$) as

$$\mathcal{P}_s\equiv\langle\mathcal{P}_s^>\rangle_<\to 1-\frac{\sqrt{s}}{\mathrm{Pe}}\int_0^\infty d\mu\ \mathcal{P}^<(\mu)\frac{I_1(\sqrt{s/\mu})}{I_0(\sqrt{s/\mu})}$$
$$=1-\frac{1}{\sqrt{\pi}\mathrm{Pe}}\int_0^\infty\ln[I_0(2\sqrt{xs})]e^{-x}dx. \tag{63}$$

The moments $\langle\epsilon^n\rangle$ can be immediately reconstructed, according to Eq. (48), from the derivatives of Eq. (63) at $s=0$. They read

$$\langle\epsilon^n\rangle=\frac{(2n-1)!!n!}{\sqrt{\pi}\mathrm{Pe}}\frac{\partial^n}{\partial z^n}\ln\left[\frac{1}{J_0(\sqrt{z})}\right]\Bigg|_{z=0}$$
$$=i\frac{\Gamma(n+1/2)[\Gamma(n+1)]^2 2^n}{2\pi^2\mathrm{Pe}}\int_C\frac{\ln[J_0(\sqrt{z})]dz}{z^{n+1}}, \tag{64}$$

where C is a close contour about $z=0$ in the complex z plane. The Cauchy integral representation (64) is useful for getting the large-n asymptote of Eq. (64). The integral is indeed saddled around the square of the first zero of the Bessel function $z_0=x_0^2$, $J_0(x_0)=0$. The saddle estimation for the integral is of the order of x_0^n, thus giving

$$\ln[\mathrm{Pe}\langle\epsilon^n\rangle]\xrightarrow[n\to\infty]{}n(3\ln[n]-3+\ln[2x_0]). \tag{65}$$

The whole expression (63) can actually be inverted. From Eq. (48) it follows indeed that the PDF $\mathcal{P}^{(\epsilon)}(\epsilon)$ of the dissipation ϵ is given by

$$\mathcal{P}^{(\epsilon)}(\epsilon)=\frac{1}{2\sqrt{\pi}}\frac{1}{2\pi i}\int_0^\infty dx\frac{e^{-x}}{x^{3/2}}\int_{0^+-i\infty}^{0^++i\infty}ds\ \mathcal{P}_s\ \exp[s\epsilon/(2x)]$$
$$=\frac{1}{\mathrm{Pe}}\int_0^\infty dz\ \ln\left[\frac{1}{I_0(z)}\right]\left[\frac{1}{\sqrt{2}\pi}\frac{{}_0F_2(1/2,1/2;\epsilon z^2/8)}{\sqrt{\epsilon}}-\frac{z}{\sqrt{\pi}}{}_0F_2(1,3/2;\epsilon z^2/8)\right], \tag{66}$$

where ${}_qF_p$ is the generalized hypergeometric function with the q parameters in the numerator and the p parameters in the denominator. The δ function at the origin arising from the unit term in Eq. (56) has not been considered in Eq. (66). The reason is that, as we shall see in a moment, the range of validity of Eq. (66) is $\epsilon\gg\mathrm{Pe}^{-2}$.

It is indeed time to clarify the limits of validity of the calculations performed. To be concrete, we shall specifically consider the pumping G^* leading to Eq. (56). The remainder of the expansion over Pe^{-1} in Eq. (56) is bounded by $\beta[I_1(\sqrt{s\beta/\alpha})\sqrt{s\alpha/\beta}+sI_1'(\sqrt{s\beta/\alpha})]/2\mathrm{Pe}^2$. Contrary to Eq. (61), we now need to consider averages of $\mu=\alpha/\beta$ with weight β. The problem can again be reduced to the calculation of a quantum-mechanical matrix element and it is found that the final result depends on t_0 as $\exp(2t_0)$. The condition for the remainder to be subdominant with respect to the terms kept in Eq. (56) is therefore that $\exp(2t_0)/\mathrm{Pe}$ should vanish as $\mathrm{Pe}\to\infty$. On the other hand, the observation time must be much larger than unity in order to attain the stationary state, i.e., $t_0\gg 1$. This condition is clearly compatible with the previous one, in the limit of large Péclet numbers, and gives the ordering $1\ll t_0\ll\ln[\mathrm{Pe}]$. The other delicate point is the criterion of applicability of Eq. (63) with respect to s. For the expansion in Eq. (56) to be meaningful, the second term should be much smaller than unity. This shows that we should require $s\ll\mathrm{Pe}^2$. The expansion (56) fails then to describe the large s tails of $\mathcal{P}_s$ and thus the smallest values of ϵ in the respective PDF (the relation between large s and small ϵ is direct since the decay of the generating function is relatively slow).

Another simple approximation is, however, available for the high-s limit. The trajectories contributing to $\langle\exp(-sQ)\rangle$ at large $s\gg\mathrm{Pe}^2$ are clearly those where Q is small. From the definition (49) it follows that, for this to happen, $W^2(t)$ should remain small all the time. The quantity $W^2\int^t W^{-2}$ is in this case $O(1)$ and the argument in G is small on account of the $1/\mathrm{Pe}^2$ factor. For the trajectories relevant at high s, it is thus possible to approximate Q by $G[0]/\mathrm{Pe}^2\int W^2$. The Laplace transform $\mathcal{P}_s$ reduces then to

$$\mathcal{P}_s\equiv\left\langle\exp\left[-\frac{sG[0]}{\mathrm{Pe}^2}\int_0^T dtW^2(t)\right]\right\rangle$$
$$=e^{-T/4}\langle\delta(\eta)|\exp\{-T\hat{H}(-\eta;sG[0]/\mathrm{Pe}^2)\}|e^{-\eta/2}\rangle, \tag{67}$$

where $\eta(t)\equiv\int_0^t\sigma(t)$ and $\hat{H}$ is the same as for Eq. (58). The matrix element (67) actually coincides identically with Eq. (58) when t_0 is replaced by T, η by $-\eta$, and s by $sG[0]/\mathrm{Pe}^2$. Using Eq. (60), one can then easily get the final answer

$$\mathcal{P}_s\to\exp\left[-\frac{\sqrt{sG[0]}}{\mathrm{Pe}}\right],\quad s\gg\mathrm{Pe}^2. \tag{68}$$

Inverting Eq. (68), we can obtain the PDF of ϵ at the small values $\epsilon\ll\mathrm{Pe}^{-2}$. (Falkovich has informed us that the expression for the PDF's origin can be derived through the adiabatic conjecture suggested recently [24].) For larger values of ϵ, the PDF follows from the general formula (66). The following general behavior for the dissipation field PDF is thus obtained:

$$\mathcal{P}^{(\epsilon)}(\epsilon)\to\begin{cases}\dfrac{\sqrt{2}\mathrm{Pe}}{\pi\sqrt{G[0]}}\dfrac{1}{\sqrt{\epsilon}}\left(1-\dfrac{2\mathrm{Pe}^2}{G[0]}\epsilon\right) & \text{for } \epsilon\ll\mathrm{Pe}^{-2}\\ \dfrac{G[0]}{\sqrt{2}\pi}\dfrac{1}{\mathrm{Pe}}\dfrac{1}{\epsilon^{3/2}} & \text{for } \mathrm{Pe}^{-2}\ll\epsilon\ll1\\ G[0]\,\dfrac{\exp[-(\epsilon/\epsilon_0)^{1/3}]}{\mathrm{Pe}},\quad \epsilon_0\sim1 & \text{for } \epsilon\gg1,\end{cases}\tag{69}$$

where algebraic prefactors have not been considered in region of exponential falloff. The PDF for scalar gradients $\omega=\sqrt{\epsilon/\kappa}$ follows immediately from Eq. (69) as

$$\mathcal{P}^{\omega}(\omega)\to\begin{cases}\dfrac{L}{\pi}\left(1-\dfrac{L^2}{G[0]}\omega^2\right) & \text{for } \omega\ll1/L\\ \dfrac{G[0]}{\pi L}\dfrac{1}{\omega^2} & \text{for } 1/L\ll\omega\ll1/\sqrt{\kappa}\\ G[0]\,\dfrac{\kappa}{\mathrm{Pe}}\exp[-(|\omega|\sqrt{\kappa/\epsilon_0})^{2/3}] & \text{for } \omega\gg1/\sqrt{\kappa}.\end{cases}\tag{70}$$

Moments of the dissipation $\langle\epsilon^a\rangle$, with $a>1/2$, are all proportional to 1/Pe, in agreement with Eq. (54) valid for arbitrary pumping. Moreover, we learn from Eq. (69) that the linear scaling found for scalar differences low-order moments comes down to the dissipative range. Moments $\langle\epsilon^a\rangle$ with $-1/2<a\leq1/2$ scale indeed as Pe^{-2a}. Moments with $a<-1/2$ do not exist. From Eq. (70) it also follows that the same tendency observed for scalar differences PDF to develop a cusped structure at the origin is present. An important difference arises for the tails. The comparison between Eqs. (70) and (44) for gradients and scalar differences indicate indeed that the former decrease much more slowly. Such behavior is physically understood in terms of fluctuations of the dissipative scale as follows. Gradients can be thought of as scalar differences evaluated at the dissipative scale. This scale is, however, a dynamical quantity and fluctuates (strongly in a problem with inverse transfer like the one here). The statistics of these fluctuations are actually related precisely to those of the variable μ introduced in Eq. (57). The fluctuations of ϵ^n are therefore the product of those of scalar differences *and* those of the dissipative scale. The latter do not depend on the dimensional parameters of the problem, i.e., they are universal as follows from Eq. (60), but grow factorially with the order n. This factorial growth is precisely what shifts the dominant term in Eq. (65) from $2n\ \ln n$ to $3n\ \ln n$. It follows then that the slower decay of the tails for gradients with respect to those for scalar differences is indeed due to dynamical fluctuations of the dissipative scale.

VII. CONCLUSIONS AND DISCUSSION

We analyzed in the present paper statistics of the passive scalar advected by a one-dimensional, smooth, and fast-correlated-in-time velocity field. In spite of its extreme simplicity, the dynamics presents very interesting and surprising behaviors. We expect that many of them are quite generic and it may be interesting to look for more realistic scalar (and generally turbulent) problems where the effects discussed here could be of relevance. Thus we recall and briefly comment on the major properties of the model discussed in the paper and address, in parallel, the respective questions for future studies.

The first major property of the dynamics is *the inverse cascade of the scalar*. The inverse cascade is a consequence of compressibility, but in a subtle way: In [25] we considered indeed a generalized smooth d-dimensional model having the degree of compressibility as a free parameter. A transition between inverse and direct cascades is observed there: If $d>4$, the cascade is always direct, independently of the degree of compressibility; if the latter is small enough, the cascade is direct again; otherwise, it is inverse. It might also be interesting to look from the same dynamical point of view used here at the critical dimension appearing in the passive vector problem discussed in [26] and at the role of compressibility for the direction of energy transfer in other turbulent situations, e.g., 2D Navier-Stokes and turbulence in magnetohydrodynamics.

Dynamically, the inverse cascade is associated with the fact that the Lyapunov exponent for Lagrangian trajectories is negative. This implies that, *in the case of inverse cascade, contraction of Lagrangian trajectories is typical and stretching is a relatively rare event.* This is precisely the opposite of the canonical picture associated with the direct cascade. Rare trajectories (contracting or stretching if the Lyapunov exponent is positive or negative, respectively) are responsible for the intermittent part of the problem. This definitely confirms the picture that contractive trajectories play a crucial role in the intermittence in incompressible turbulent transport.

A negative Lyapunov exponent leads to the third observation emerging from the analytical study of the model: *The Gaussian statistics of the scalar is established at scales larger than the scale of the pumping, while strong intermittence is present at small scales.* The small-scale intermittence found here is of the Burgers type, i.e., all scalar difference integer moments scale linearly $\sim r/L$. This result gives a definite ground to speculate that, generally, the stronger the degree of compressibility (the role of contracting trajectories is then enhanced), the more intermittence downscaling of the pumping is observed. Note with respect to the Gaussian nature that the inverse cascade of energy in 2D Navier-Stokes turbulence is also Gaussian according to the numerics [27]. It raises yet another conjecture of a certain generality of the Gaussian feature for a great variety of direct cascades.

The equation for the scalar difference PDF, derived directly from the dynamics in Sec. V, shows that *no dissipative anomaly is present* in the model. In the general one-dimensional model introduced in [14], $\gamma=1$ is a natural threshold for the inverse cascade. At $\gamma>1$ the cascade is direct, while it is inverse at $\gamma<1$. Most probably this means that there is no dissipative anomaly (the equation for the PDF of the scalar difference could be closed) for $\gamma<1$. However, it is not yet clear how to derive this consistently (as it is derived here for the smooth limit $\gamma=0$). Note that the general question of the role of the dissipative anomaly in the passive scalar physics is not yet resolved in any sense (some suggestions on this point may be found in [6,19,28]). One more unresolved question is how the dissipative anomaly (if any) affects the anomalous scaling behavior of structure functions of integer (and generally all) orders.

An important result for the issue of convection-diffusion interplay is that *the effective dissipative scale*, which sets the crossover scale between the scalar difference $2n$th moment in the convective range and the dissipation field nth moment, *is a strongly fluctuating quantity*, growing factorially with n. To make this statement, we have analytically calculated in Sec. VI the dissipation field PDF exploiting a different scale-separation procedure, which is also used in Appendix B to describe the long-time dynamics of the pair-correlation function of the scalar gradients. The procedure is likely to be a general tool for resolving the problem in other situations. Particularly, a slight (matrix) modification of the method should be relevant for the Kraichnan d-dimensional incompressible smooth model.

Finally, the eddy-diffusivity operator is Hermitian for the Kraichnan model, where the random velocity is incompressible and fast. *The Hermitian nature is lost when compressible flow is considered*. Generally, it might be interesting to look at scalar transport (and particularly at flow with finite correlation times) from the point of view of non-Hermitian nature. This issue is the subject of very recent interest in the field of disordered systems [29–32]. The specific object of interest for scalar transport is the resolvent $\mathcal{R}(t;x,y)$, describing the probability to find a Lagrangian separation equal to x at time t, having initially been equal to y. One may then want to study the distribution in the complex plane of the poles of its Laplace transform. A width of the poles' domain along the imaginary direction might be a characteristic of the trapping degree.

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APPENDIX A: DYNAMICAL FORMULATION BY FIELD FORMALISM

Let us consider the gradient field $\omega(t,x)$ satisfying the equation of motion (4) with the velocity $u(t,x)$ having Gaussian statistics and correlation function $\langle u(t,x)u(t',x')\rangle = V(t-t',x-x')$. The generating functional $\mathcal{F}(\lambda)$ for its simultaneous correlation functions can be written in the form of a functional integral (see [33,34])

$$\mathcal{F}(\lambda)=\int \mathcal{D}p\ \mathcal{D}\omega\ \mathcal{D}u\ \exp\left\{\frac{1}{2}(p|\chi''|p)-\frac{1}{2}(u|V^{-1}|u) + i\int_0^T dt\ dx\ p[\partial_t\omega+\partial_x(u\omega)-\kappa\partial_x^2\omega] + \int dx\ \lambda(x)\omega(T,x)\right\}. \tag{A1}$$

Here $(u|V^{-1}|u)$ denotes the diagonal matrix element of the inverse operator to $V(t-t',x-x')$ in the Hilbert space of the functions $u(t,x)$ and the pumping χ'' is the second spatial derivative of the forcing correlation function χ, defined in Eq. (2). The retarded regularization of the time derivative, $(\partial_t\omega)_n=(1/\Delta)(\omega_n-\omega_{n-1})$ does not produce any nontrivial Jacobian after derivation of Eq. (A1) from the equation of motion. $\Delta\to 0$ is the temporal slicing and $\omega_n(x)\equiv\omega(n\Delta,x)$. Performing the Gaussian integration over the field $u(t,x)$, we arrive at

$$\mathcal{F}(\lambda)=\int \mathcal{D}p\ \mathcal{D}\omega\ \exp\left\{-\frac{1}{2}(\omega\partial_x p|V|\omega\partial_x p)+\frac{1}{2}(p|\chi''|p) + i\int_0^T dt\ dx\, p(\partial_t\omega-\kappa\partial_x^2\omega)+\int dx\ \lambda(x)\omega(T,x)\right\}. \tag{A2}$$

For the case of δ-correlated-in-time velocity fields considered in this paper, we replace the general $V(t-t',x-x')$ by specific $\delta(t-t')V(x-x')$. The explicit version of Eq. (A1) then reads

$$\mathcal{F}(\lambda)=\int \mathcal{D}\ p\mathcal{D}\omega\ \exp\left(-\mathcal{S}+\int dx\ \lambda(x)\omega(T,x)\right), \tag{A3}$$

$$\mathcal{S}=\frac{1}{2}\int_0^T dt\left[\int dx\ dx'(\omega\partial_x p)(t,x)V(x-x')(\omega\partial_{x'}p)(t,x') - \int dx\ dx'\, p(t,x)\chi''\left(\frac{x-x'}{L}\right)p(t,x')\right] - i\int_0^T dt\ dx\, p(\partial_t\omega-\kappa\partial_x^2\omega). \tag{A4}$$

Despite the explicit presence of the V term in Eq. (A4), the dynamical part of the action S is covariant with respect to Galilean transformations. This leads to the important Ward identity

$$\int \mathcal{D}p\ \mathcal{D}\omega\ \exp\left(-\mathcal{S}+\int dx\ \lambda(x)\omega(T,x)\right)\times\prod_j\left(\int dx\ \omega(t_j,x)\partial_x p(t_j,x)\right)=0, \quad (A5)$$

where the set $\{t_j<T, j=1,2,\dots\}$ is arbitrary. To prove it, let us consider the change of variables

$$p(t,x)\to p\left(x+\int_t^T v(\tau)d\tau,t\right),$$

$$\omega(t,x)\to\omega\left(x+\int_t^T v(\tau)d\tau,t\right) \quad (A6)$$

in the functional integral (A3) with $v(\tau)$ being unconstrained. The transformation does not change the source term and it has a unit Jacobian. The variation $\delta\mathcal{S}$ of the action $\mathcal{S}$ under the transformation reads

$$\delta\mathcal{S}=-i\int_0^T dt\, v(t)\left(\int dx\ \omega(t,x)\partial_x p(t,x)\right). \quad (A7)$$

The change of variables does not change the value of the integral. Thus all the functional derivatives of Eq. (A3) over $v(t)$ are equal to zero. Taking into account Eq. (A7), one finally arrives at the Ward identity (A5).

All expressions written above are applicable for any short-correlated velocity field. Now let us narrow the consideration to the case of smooth velocity field (3) with $\gamma=0$. One gets

$$\mathcal{F}(\lambda)=\int \mathcal{D}p\ \mathcal{D}\omega\ \exp\left(-\mathcal{S}+\int dx\ \lambda(x)\omega(T,x)\right), \quad (A8)$$

$$\mathcal{S}=\frac{1}{2}\int_0^T dt\left[-\int dx\ dx'(\omega\partial_x p)(t,x)(x-x')^2(\omega\partial_{x'}p)\times(t,x')-\int dx\ dx'\,p(t,x)\chi''\left(\frac{x-x'}{L}\right)p(t,x')\right] \quad (A9)$$

$$-i\int_0^T dt\ dx\, p(\partial_t\omega-\kappa\partial_x^2\omega), \quad (A10)$$

where the V_0 term was dropped due to Eq. (A5). The goal is to reduce the problem to averaging of a functional of one random-in-time scalar field, analogously to the random matrix description used in the two-dimensional [5,35] and generally d-dimensional cases. However, in the present case, compressibility calls for further evaluation. Let us perform the following change of variables in Eq. (A8):

$$\{p(t,x),\omega(t,x)\}\to\left\{p\left(t,x+i\int_t^T d\tau\int dy\, y^2\partial_y p(t,y)\omega(t,y)\right),\right.$$
$$\left.\omega\left(t,x+i\int_t^T d\tau\int dy\ y^2\partial_y p(t,y)\omega(t,y)\right)\right\}, \quad (A11)$$

which can be considered as the field-dependent and time-dependent homogeneous spatial translation leaving the source term intact. Analytic continuation of the fields, which makes the spatial arguments real, is assumed. The action (A9) gets the following form being rewritten in the new variables:

$$\mathcal{S}=\frac{1}{2}\int_0^T dt\left[2\left(\int dx\, x(\omega\partial_x p)(t,x)\right)^2-\int dx\ dx'\,p(t,x)\chi''\left(\frac{x-x'}{L}\right)p(t,x')\right]-i\int_0^T dt\ dx\, p(\partial_t\omega-\kappa\partial_x^2\omega). \quad (A12)$$

The transformation (A11) has a nontrivial Jacobian

$$\mathcal{D}p\mathcal{D}\omega\to\mathcal{D}p\mathcal{D}\omega\mathcal{J}(p,\omega) \quad (A13)$$

that depends on regularization. The regularization is fixed by the requirement for the temporal δ function from the correlation function of velocities to appear as a result of narrowing of an even function of temporal argument. In addition, the correctly regularized action (A12) should reproduce the respective correlation function in the limit $T\to 0$:

$$p_n(x)\to p_n\left(x+\frac{i}{2}\Delta\int dy\ y^2\partial_y p_n(y)\omega_n(y)+i\Delta\sum_{m=n+1}^{N}\int dy\ y^2\partial_y p_m(y)\omega_m(y)\right) \quad (A14)$$

for $\omega_n(x)$. The Jacobian $\mathcal{J}(p,\omega)$ can be computed easily

$$\mathcal{J}(p,\omega)=\exp\left(-i\int dt\ dx\, x\omega\partial_x p\right). \quad (A15)$$

The exponential of the term $\int_0^T dt(\int dx\ x\omega\partial_x p)^2$ in the action (A12) multiplied by the Jacobian (A15) can be represented by means of averaging the exponential of $i\int_0^T dt\ \sigma\int dx\ x\ \omega\partial_x p$ over the auxiliary field $\sigma(t)$. The Gaussian measure of averaging over σ is defined in Eq. (21). It results in the following representation for the generating functional $\mathcal{F}(\lambda)$:

$$\mathcal{F}(\lambda)=\int \mathcal{D}p\ \mathcal{D}\omega\ \mathcal{D}f\ \mathcal{D}\sigma\ \exp\left[-\mathcal{S}_\sigma-\frac{1}{2}(f|\chi^{-1}|f)\right]\times\exp\left[i\int_0^T dt\ dx\, p[\partial_t\omega-\sigma\partial_x(x\omega)-\kappa\partial_x^2\omega-\partial_x f]+\int dx\ \lambda(x)\omega(T,x)\right]. \quad (A16)$$

Integration over $p(t,x)$ gives the reduced equation of motion in the reference frame comoving with a fluid particle

$$\partial_t\omega-\sigma\partial_x(x\omega)-\kappa\partial_x^2\omega=\partial_x f, \tag{A17}$$

averaged with respect to $f(t,x)$ and $\sigma(t)$ with the weights given by the first line of (A16). It should be stressed that this equivalence holds only at the level of simultaneous correlation functions.

APPENDIX B: LONG-TIME BEHAVIOR OF THE PAIR-CORRELATION FUNCTION OF GRADIENTS

In the present appendix we will find the long-time asymptotic of the pair-correlation function of the scalar gradients $\Omega(t,x)$ that obeys Eq. (5). We will model here the infrared cut-off of the velocity correlations by

$$S(x)=x^2,\quad x<L_u,\quad S(x)=L_u^2,\quad x>L_u. \tag{B1}$$

The Laplace transform of Eq. (5) has the form

$$s\Omega_s-\partial_x^2[S(x)+2\kappa]\Omega_s+\chi''(x)/s=0. \tag{B2}$$

The solution of Eq. (B2) can be written in terms of the Green's function $\mathcal{G}_s(x,x')$ [which is taken to be even in x, $\mathcal{G}_s(x,-x')=\mathcal{G}_s(x,x')$] as

$$\Omega_s(x)=-\frac{1}{s}\int_0^\infty \mathcal{G}_s(x,x')\chi''(x')dx',$$

$$s\mathcal{G}_s-\partial_x^2[S(x)+2\kappa]\mathcal{G}_s=\delta(x-x'). \tag{B3}$$

The desired long-time asymptotic corresponds to the smallest s and our aim is to find the singular structure of $\Omega_s(x)$ at small s and $x\gg\sqrt{\kappa}$. In what follows we will then neglect all logarithmic contributions such as $s\ \ln L_u$ and $s\ \ln x$. We will also neglect the smallest x' $(x'\ll\sqrt{\kappa})$ contribution in the integral (B3) going to zero in the limit of small diffusivity. Therefore, $\mathcal{G}_s(x,x')$ should be studied at $\sqrt{\kappa}\ll x'\ll L_u$ and in all the allowed domains with respect to x.

Making the substitution $\mathcal{G}_s(x,x')=g(x,x')/[S(x)+2\kappa]$, one gets

$$\frac{s}{S(x)+2\kappa}g-\partial_x^2 g=\delta(x-x'). \tag{B4}$$

The solution of Eq. (B4) in the domain of the largest $x, x>L_u>x'\gg\sqrt{\kappa}$, can thus be written as

$$g(x,x')=A\exp\left(-\frac{\sqrt{s}}{L_u}(x-L_u)\right). \tag{B5}$$

Note, however, that only the first two terms of the expansion (B5) in s will be required for matching hereafter. If x is smaller than L_u, but still larger than a separation scale x_0, one can omit the first term on the right-hand side of Eq. (B4) and write the general solution of Eq. (B4) in the intermediate domain, at $L_u>x\gg\sqrt{\kappa}$,

$$g(x,x')=\begin{cases}F_1+xF_2, & x<x'\\ D_1+xD_2, & x>x'.\end{cases} \tag{B6}$$

Here, the constants F_1,F_2,D_1,D_2 should be fixed via inner matching at $x=x'$ and outer matchings at L_u and x_0. The value of x_0, restricted by $\sqrt{\kappa}\ll x_0\ll\sqrt{\kappa}/s$, defines the upper bound for the scales at which one cannot omit the first term of Eq. (B4) anymore. However, the point is that at $x<x_0<x'$ an actual dependence of g on x_0 has disappeared and we should not locate x_0 explicitly to get the asymptotic behavior of $g(x,x')$. Indeed, integrating Eq. (B4), we get

$$\partial_x g(x)=s\int_0^x\frac{g(z)dz}{S(x)+2\kappa}, \tag{B7}$$

where the argument x' is omitted and it is already taken into account that $g(x)$ is even: $\partial_x g(x)|_{x=0}=0$. Equation (B7) shows that the x derivative changes sufficiently at the diffusive scale, being nonetheless small there. This means that the function $g(x)$ at $x\lesssim x_0$ is a constant in the leading (zero) order in s. The subleading (first order in s) term is then fixed by Eq. (B7) taken at $x=x_0$. All together this gives the following expression for g at $x<x'$ with the desirable (for the forthcoming matching) accuracy:

$$g\approx g_0\left(1+sx\int_0^{x_0}\frac{dz}{S(x)+2\kappa}\right)\approx g_0\left(1+s\frac{\pi x}{2\sqrt{2\kappa}}\right), \tag{B8}$$

where, evaluating the integral in Eq. (B8), we used that $\sqrt{\kappa}\ll x_0$. Matching Eqs. (B5) and (B6) and Eq. (B8), we get

$$D_1+x'D_2=F_1+x'F_2,\quad F_2-D_2=1\quad \text{at}\ x=x',$$

$$D_1+L_uD_2=A,\quad D_2=-\frac{\sqrt{s}}{L_u}A\quad \text{at}\ x=L_u,$$

$$F_1+x_0F_2=g_0,\quad F_2=sg_0\frac{\pi}{2r_d}\quad \text{at}\ x=x_0. \tag{B9}$$

Keeping s very small, while the ratios $s/\sqrt{\kappa}$ and $\sqrt{s}/L_u$ are finite, one derives from Eq. (B9)

$$F_1=g_0=\frac{1+x'\sqrt{s}/L_u}{s\pi/(2r_d)+\sqrt{s}/L_u},\quad F_2=\frac{sg_0\pi}{2\sqrt{2\kappa}},$$

$$D_1=A=\frac{2\sqrt{2\kappa}+x's\pi}{s\pi+2\sqrt{2\kappa s}/L_u},\quad D_2=-\frac{\sqrt{s}}{L_u}A. \tag{B10}$$

It is worth noting that the explicit value of x_0 does not enter Eq. (B10) and the scale separation procedure is indeed justified. Substituting Eq. (B10) into Eq. (B6) $(x'<L_u)$, one gets

$$\Omega_s(x)=\frac{1}{S(x)+2\kappa}\left[\frac{\chi(x)}{s}-\frac{\chi(0)}{s+\alpha\sqrt{s}}\right],\quad \alpha=\frac{2\sqrt{2\kappa}}{\pi L_u}, \tag{B11}$$

and the inverse Laplace transform

$$\Omega(t,x)=\frac{1}{S(x)+2\kappa}\left[\chi(x)-\chi(0)\frac{2}{\pi}\int_0^\infty dp\,\frac{\exp(-\alpha^2p^2t)}{p^2+1}\right]. \tag{B12}$$

Expression (B12) states that, at the times less than $\mathcal{T}_{L_u}\sim L_u^2/\kappa$, the correlation function coincides with the quasistationary limit given by Eq. (7). At the largest times, $\Omega(t,x)$ approaches the asymptotic value

$$\frac{1}{S(x)+2\kappa}\chi(x) \tag{B13}$$

diffusively, $\chi(x)-\Omega[S(x)+2\kappa]=C\to\chi(0)(\alpha^2\pi t)^{-1/2}$.

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